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The quark form factor at higher orders

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Abstract: We study the electromagnetic on-shell form factor of quarks in massless perturbative QCD. We derive the complete pole part in dimensional regularization at three loops, and extend the resummation of the form factor to the next-to-next-to-leading contributions. These results are employed to evaluate the infrared finite absolute ratio of the time-like and space-like form factors up to the fourth order in the strong coupling constant. Besides for the pole structure of higher-loop QCD amplitudes, our new contributions to the form factor are also relevant for the high-energy limit of massive gauge theories like QED. The highest-transcendentality component of our results confirms a result recently obtained in $\mathcal{N} = 4$ Super-Yang-Mills theory.

Keywords: Extended Supersymmetry, Electromagnetic Processes and Properties, QCD.

Contents

1. Introduction

The electromagnetic form factor of quarks is a quantity of considerable interest in Quantum Chromodynamics (QCD) and in gauge theories in general. At high photon virtualities Q^2 this quantity receives double logarithmic corrections of infrared and collinear origin [1], which take the form of double poles in dimensional regularization for the case of massless on-shell quarks studied in the present article. These contributions can be resummed by evolution equations in Q^2 based on universal factorization properties of the amplitude in the relevant kinematic limit, resulting in the well-known exponentiation of the form factor [2–4]. So far perturbative calculations have been performed up to two loops for both the massless on-shell case [5, 6] and heavy quarks [7]. Accordingly, the exponentiation has been studied up to the next-to-leading (NL) contributions [8 – 10].

Higher-order corrections to the quark form factor are not only of general interest in quantum field theory, but also relevant for practical applications, as this quantity contributes to phenomenologically important processes. Research in the past years has yielded dramatic progress in next-to-next-to-leading order (NNLO) perturbative calculations, see, for example, ref. [11] and numerous references therein. This progress also led to further investigations of the general structure of amplitudes and cross sections at higher loop-orders, which in turn further stimulated the interest in all-order resummations. Consequently, the intimate connection between resummation and perturbative results at multiple loops has become much more prominent [12 – 14].

Very recently, we have presented the first complete calculation of the third-order corrections to a hard-scattering observable depending on a dimensionless variable, the structure

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function F_2 in photon-exchange deep-inelastic scattering [15]. After exploring the consequences of that result for the soft-gluon threshold resummation in ref. [16], we here present its implications on the quark form factor (from now on always referring to the massless on-shell case, if not explicitly indicated otherwise) and its resummation. We are able, after extending the two-loop form factor beyond the previous order ϵ^0 in dimensional regularization, to derive the complete series of poles, $\epsilon^{-6} \ldots \epsilon^{-1}$, at three loops. These terms in turn provide the coefficients required to extend the exponentiation of the form factor to the next-to-next-to-leading (NNL) contributions which we work out explicitly.

This article is organized as follows: In section 2 we address the resummation of the quark form factor. We briefly recall the evolution equation and its solution, and present the explicit expansion up to four loops in terms of two perturbative functions, $A(\alpha_s)$ (known up to three loops from the NNLO splitting functions [17, 18]) and $G(\alpha_s, \epsilon)$. In section 3 we extend the two-loop form factor to order ϵ^2 and extract the pole terms at three loops from our structure-function calculation [15]. These results are employed to extend the first- and second-order parts of the resummation function G to higher orders in ϵ, and to derive the leading-ϵ term at the third order in the strong coupling. Some first implications of these results are discussed in section 4. Here we extend the ratio of the time-like and space-like form factors [9] to the fourth order in α_s, compare to a recent result for $\mathcal{N} = 4$ Super-Yang-Mills theory [19] and indicate applications on the infrared structure of massive gauge theories [20, 21]. We briefly summarized our results in section 5. A few technical details for the solution of the evolution equations in section 2 can be found in appendix A. Finally the break-up of the (ϵ-extended) two-loop form factor into its Feynman diagrams is presented in appendix B.

2. The resummation of the quark form factor

The subject of our study are the QCD corrections to the $\gamma^* qq$ (or $\gamma^* q\bar{q}$) vertex, where γ^* denotes a space-like (or time-like) photon with virtuality Q^2, and $q/\bar{q}$ a massless external quark/antiquark. Until section 4 we will focus on the space-like case, thus the relevant amplitude is

$$\Gamma_\mu = \mathrm{i} e_\mathrm{q} \, (\bar{u} \, \gamma_\mu \, u) \, \mathcal{F}(\alpha_s, Q^2) \,, \tag{2.1}$$

where the scalar function $\mathcal{F}$ on the right-hand side is the space-like quark form factor. This quantity can be calculated order by order in the strong coupling constant α_s and, as mentioned above, is so far known to two loops [5, 6]. $\mathcal{F}$ is gauge invariant, but divergent. As usual we work in dimensional regularization with $D = 4 - 2\epsilon$, thus these divergences show up as poles ϵ^{-k} in the present article.

The exponentiation of the form factor, which extends beyond the resummation of renormalization group logarithms, is achieved by solving the well-known evolution equations [3, 8, 4, 9, 10]

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}\left(\alpha_s, \frac{Q^2}{\mu^2}, \epsilon\right) = \frac{1}{2} K(\alpha_s, \epsilon) + \frac{1}{2} G\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) . \tag{2.2}$$

Here μ represents the renormalization scale, and the functions G and K are subject to the renormalization group equations [3]

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_{\rm s}, \epsilon) \frac{\partial}{\partial \alpha_{\rm s}}\right) G\left(\frac{Q^2}{\mu^2}, \alpha_{\rm s}, \epsilon\right) = A(\alpha_{\rm s}), \tag{2.3}$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_{\rm s}, \epsilon) \frac{\partial}{\partial \alpha_{\rm s}}\right) K(\alpha_{\rm s}, \epsilon) = -A(\alpha_{\rm s}). \tag{2.4}$$

All infrared singularities are collected by the scale-independent function K, which in the $\overline{\rm MS}$ scheme consists of a series of poles in ϵ. The function G, on the other hand, is finite for $\epsilon \to 0$ and includes all dependence on the scale Q^2. The renormalization properties of G and K are both governed by the same anomalous dimension A, because the sum of G and K is a renormalization group invariant. This quantity is given by a power expansion in the strong coupling, for which we use the convention (also employed for all other expansions in $\alpha_{\rm s}$ throughout this article)

$$A(\alpha_{\rm s}) = \sum_{i=1}^{\infty} \left(\frac{\alpha_{\rm s}}{4\pi}\right)^i A_i \equiv \sum_{i=1}^{\infty} a_{\rm s}^{\,i} A_i . \tag{2.5}$$

In fact, the anomalous dimension A also occurs in many other circumstances, for instance as the coefficient of the $1/(1-x)_+$ contribution to the Altarelli-Parisi quark-quark splitting function and as the anomalous dimension of a Wilson line with a cusp [22].

As already indicated by the argument of the beta function, the solution of eqs. (2.3) and (2.4) requires the running coupling in D dimensions. Following refs. [10, 23] we define $\bar{a}(\lambda, a_s, \epsilon)$, where λ is a dimensionless ratio of scales like $\lambda = Q^2/\mu^2$. The resummation of the NNL contributions to the form factor requires the scale dependence of $\bar{a}$ to NNLO accuracy [24, 25] (see the discussion at the end of this section), obtained by solving

$$\lambda \frac{\partial}{\partial \lambda} \bar{a}(\lambda, a_s, \epsilon) = -\epsilon\, \bar{a}(\lambda, a_s, \epsilon) - \beta_0\, \bar{a}^2(\lambda, a_{\rm s}, \epsilon) - \beta_1\, \bar{a}^3(\lambda, a_{\rm s}, \epsilon) - \beta_2\, \bar{a}^4(\lambda, a_{\rm s}, \epsilon) \tag{2.6}$$

with the boundary condition $\bar{a}(1, a_{\rm s}, \epsilon) = a_{\rm s}$. Extending the result of ref. [23] by one order, this solution is given by

$$\begin{aligned} \bar{a}(\lambda, a_{\rm s}, \epsilon) = {} & \frac{a_{\rm s}}{X}\left\{1 - \epsilon \frac{\beta_1}{\beta_0^2} \frac{\ln X}{X}\right\} - \frac{a_{\rm s}^2}{X^2}\left\{\frac{\beta_1}{\beta_0}(\ln X + Y)\right\} + \\ & + \frac{a_{\rm s}^3}{X^3}\left\{\frac{\beta_1^2}{\beta_0^2}\frac{3}{2}\ln^2 X \left(1 + Y + \frac{1}{4}Y^2\right) + \right. \\ & \left. + \frac{\beta_2}{\beta_0} \ln X \left(\frac{1}{6}(3+Y)(1-X) - 1 - Y - \frac{1}{3}Y^2\right)\right\} + \mathcal{O}\left(a_{\rm s}^4\right), \end{aligned} \tag{2.7}$$

where we have used the abbreviations

$$X = 1 - a_{\rm s}\frac{\beta_0}{\epsilon}(\lambda^{-\epsilon} - 1), \qquad Y = \frac{\epsilon(1-X)}{a_{\rm s}\beta_0}. \tag{2.8}$$

With eq. (2.7) for the running coupling, eq. (2.3) can now be solved to the required accuracy,

$$G\left(\frac{Q^2}{\mu^2}, \alpha_{\rm s}, \epsilon\right) = G\left(1, \bar{a}\left(\frac{Q^2}{\mu^2}, a_{\rm s}, \epsilon\right), \epsilon\right) + \int\limits_{Q^2/\mu^2}^{1} \frac{d\lambda}{\lambda}\, A(\bar{a}(\lambda, a_{\rm s}, \epsilon))\,. \tag{2.9}$$

The perturbative expansion of the boundary condition $G(1, \bar{a}, \epsilon)$ can be derived by comparison to the fixed-order results for the form factor.

After recursively determining (see, e.g., ref. [10] for details) the scale-independent counter-term function K from eq. (2.4), the resummed quark form factor reads

$$\ln \mathcal{F}\left(\alpha_{\rm s}, \frac{Q^2}{\mu^2}, \epsilon\right) = \frac{1}{2} \int\limits_{0}^{Q^2/\mu^2} \frac{d\xi}{\xi} \left(K(\alpha_{\rm s}, \epsilon) + G(1, \bar{a}(\xi, a_s, \epsilon), \epsilon) + \int\limits_{\xi}^{1} \frac{d\lambda}{\lambda} A(\bar{a}(\lambda, a_{\rm s}, \epsilon)) \right) \tag{2.10}$$

with the boundary condition $\mathcal{F}(\alpha_s, 0, \epsilon) = 1$ [9]. After expanding the D-dimensional coupling according to eq. (2.7), $\ln \mathcal{F}$ exhibits double logarithms of Q^2/μ^2 and double poles in ϵ, which are generated by the two integrations. In addition the integral over the anomalous dimension A leads to terms which are independent of the outer integration variable ξ. These logarithmic singularities at $\xi = 0$ are canceled by the function K order by order in the perturbative expansion.

The well-known relation (2.10) can be employed either for a direct evaluation of the form factor due to the analyticity in D dimensions [10] or, by means of finite-order expansions and matching, for predictions of perturbative results at higher orders. Here we will focus on the latter issue. In particular, we will derive explicit results at three and four loops. This is done by performing the integrations in eq. (2.10) after inserting the perturbative expansions of all quantities. The resulting integrals can be evaluated using algorithms for the evaluations of nested sums [26, 27]. Some technical details for this step are given in appendix A, where eqs. (A.4)–(A.7) represent sample types of relevant integrals. Further details may also be found in ref. [10].

It is convenient to express the loop-expanded form factor in terms of the bare (unrenormalized) coupling $\alpha_{\rm s}^{\rm b}$ instead of the renormalized coupling $\alpha_{\rm s}$ as in eq. (2.10). The couplings $\alpha_{\rm s}^{\rm b}$ and $\alpha_{\rm s}$ are related by

$$\alpha_{\rm s}^{\rm b} = Z_{\alpha_{\rm s}}\, \alpha_{\rm s}\,, \tag{2.11}$$

with the renormalization constant $Z_{\alpha_{\rm s}}$ in the $\overline{\rm MS}$ scheme given by

$$Z_{\alpha_{\rm s}} = 1 - \frac{\beta_0}{\epsilon} a_{\rm s} + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{1}{2}\frac{\beta_1}{\epsilon}\right) a_{\rm s}^{\,2} - \left(\frac{\beta_0^3}{\epsilon^3} - \frac{7}{6}\frac{\beta_1\beta_0}{\epsilon^2} + \frac{1}{3}\frac{\beta_2}{\epsilon}\right) a_{\rm s}^{\,3}\,, \tag{2.12}$$

and also the bare expansion parameter normalized as $a_{\rm s}^{\rm b} = \alpha_{\rm s}^{\rm b}/(4\pi)$. The perturbative expansion of the bare (unrenormalized) quark form factor then reads

$$\mathcal{F}^{\rm b}(\alpha_s^{\rm b}, Q^2) = 1 + \sum_{l=1}^{\infty} (a_s^{\rm b})^l \left(\frac{Q^2}{\mu^2}\right)^{-l\epsilon} \mathcal{F}_l\,. \tag{2.13}$$

In terms of the i-th order parameters A_i in eq. (2.5) and the corresponding functions $G_i(\epsilon)$, the expansion coefficients up to four loops read

$$\mathcal{F}_1 = -\frac{1}{2}\frac{1}{\epsilon^2}A_1 - \frac{1}{2}\frac{1}{\epsilon}G_1\,, \tag{2.14}$$

$$\mathcal{F}_2 = \frac{1}{8}\frac{1}{\epsilon^4}A_1^2 + \frac{1}{8}\frac{1}{\epsilon^3}A_1(2G_1 - \beta_0) + \frac{1}{8}\frac{1}{\epsilon^2}(G_1^2 - A_2 - 2\beta_0 G_1) - \frac{1}{4}\frac{1}{\epsilon}G_2\,, \tag{2.15}$$

$$\begin{aligned}\mathcal{F}_3 = &-\frac{1}{48}\frac{1}{\epsilon^6}A_1^3 - \frac{1}{16}\frac{1}{\epsilon^5}A_1^2(G_1 - \beta_0) - \frac{1}{144}\frac{1}{\epsilon^4}A_1(9G_1^2 - 9A_2 - 27\beta_0 G_1 + 8\beta_0^2) - \\ &-\frac{1}{144}\frac{1}{\epsilon^3}(3G_1^3 - 9A_2G_1 - 18A_1G_2 + 4\beta_1 A_1 - 18\beta_0 G_1^2 + 16\beta_0 A_2 + 24\beta_0^2 G_1) + \\ &+\frac{1}{72}\frac{1}{\epsilon^2}(9G_1G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2) - \frac{1}{6}\frac{1}{\epsilon}G_3\,,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathcal{F}_4 = &\frac{1}{384}\frac{1}{\epsilon^8}A_1^4 + \frac{1}{192}\frac{1}{\epsilon^7}A_1^3(2G_1 - 3\beta_0) + \frac{1}{1152}\frac{1}{\epsilon^6}A_1^2(18G_1^2 - 18A_2 - 72\beta_0 G_1 + 41\beta_0^2) + \\ &+\frac{1}{576}\frac{1}{\epsilon^5}A_1(6G_1^3 - 18A_2G_1 - 18A_1G_2 + 8\beta_1 A_1 - 45\beta_0 G_1^2 + 41\beta_0 A_2 + \\ &\qquad + 82\beta_0^2 G_1 - 18\beta_0^3) + \\ &+\frac{1}{1152}\frac{1}{\epsilon^4}(3G_1^4 - 18A_2G_1^2 + 9A_2^2 - 72A_1G_1G_2 + 32A_1A_3 + 64\beta_1 A_1 G_1 - \\ &\qquad - 36\beta_0 G_1^3 + 100\beta_0 A_2 G_1 + 228\beta_0 A_1 G_2 - 48\beta_0\beta_1 A_1 + 132\beta_0^2 G_1^2 - \\ &\qquad - 108\beta_0^2 A_2 - 144\beta_0^3 G_1) + \\ &+\frac{1}{288}\frac{1}{\epsilon^3}(-9G_1^2G_2 + 8A_3G_1 + 9A_2G_2 + 24A_1G_3 - 3\beta_2 A_1 + 12\beta_1 G_1^2 - 9\beta_1 A_2 + \\ &\qquad + 66\beta_0 G_1 G_2 - 27\beta_0 A_3 - 48\beta_0\beta_1 G_1 - 108\beta_0^2 G_2) + \\ &+\frac{1}{96}\frac{1}{\epsilon^2}(3G_2^2 + 8G_1G_3 - 3A_4 - 4\beta_2 G_1 - 12\beta_1 G_2 - 36\beta_0 G_3) - \frac{1}{8}\frac{1}{\epsilon}G_4\,.\end{aligned} \tag{2.17}$$

The three- and four-loop relations (2.16) and (2.17) are new results of the present article. Recall that eqs. (2.14)–(2.17) directly refer to the bare form factor. The corresponding renormalized results can be derived with the help of eqs. (2.11) and (2.12).

The ϵ^0 term of G_1, together with β_0 and the lowest-order anomalous dimension A_1, specify the two most singular terms ϵ^{-2n} and ϵ^{-2n+1} to all orders α_s^n. Likewise, if (besides two more contributions to G_1) also the leading term of G_2 and the NLO quantities β_1 and A_2 are known, the resummation fixes the first four leading poles at each order. This has been the status up to now, referred to as the next-to-leading (NL) contributions in section 1. In the next section, we will present the leading term of G_3 and the corresponding higher coefficients in the ϵ-expansions of G_2 and G_3. Together with β_2 (as indicated before eq. (2.6)) and our recent result for A_3 [17], these results provide the NNL terms at all orders, especially fixing the ϵ^{-4} and ϵ^{-3} poles in eq. (2.17).

3. Fixed-order results and resummation coefficients

We now turn to the extraction of the quark form factor up to order α_s^3 from our third-order computation of the deep-inelastic structure functions [15]. As also discussed in refs. [17, 18], the calculation has been performed via forward Compton amplitudes and the optical theorem. The cuts of the corresponding diagrams always include real-emission contributions,

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thus the purely virtual form-factor part cannot be directly read off at this level. Nevertheless we can reconstruct the form factor from our results, except (as explained below) at the highest power of ϵ which was consistently kept in the calculations. Consequently, we can derive all $1/\epsilon$ pole terms at order α_s^3, since the forward Compton amplitudes have been computed to order ϵ^0 for ref. [15].

Our starting point for the determination of the form factor is the unrenormalized (and unfactorized) partonic structure function $F^{\rm b}$ for $\gamma^* q \to X$ in the limit $x \to 1$, where x denotes the partonic Bjorken variable. Using the end-point properties of the harmonic polylogarithms [28] in which these results are expressed, we remove all regular contributions and only retain the singular pieces proportional to $\delta(1-x)$ and the +-distributions at order α_s^n,

$$\mathcal{D}_k = \left[\frac{\ln^k(1-x)}{1-x}\right]_+ , \qquad k = 1, \ldots 2n-1\,. \tag{3.1}$$

The resulting expressions are then compared to the general structure of the n-th order contribution $F_n^{\rm b}$ in terms of the l-loop form factors $\mathcal{F}_l$ and the corresponding pure real-emission parts $\mathcal{S}_l$,

$$\begin{aligned}
F_0^{\rm b} &= \delta(1-x)\\
F_1^{\rm b} &= 2\mathcal{F}_1\,\delta(1-x) + \mathcal{S}_1\\
F_2^{\rm b} &= 2\mathcal{F}_2\,\delta(1-x) + (\mathcal{F}_1)^2\,\delta(1-x) + 2\mathcal{F}_1\mathcal{S}_1 + \mathcal{S}_2\\
F_3^{\rm b} &= 2\mathcal{F}_3\,\delta(1-x) + 2\mathcal{F}_1\mathcal{F}_2\,\delta(1-x) + 2\mathcal{F}_2\mathcal{S}_1 + 2\mathcal{F}_1\mathcal{S}_2 + \mathcal{S}_3\,.
\end{aligned} \tag{3.2}$$

The x-dependence of the factors $\mathcal{S}_k$ is given by the D-dimensional +-distributions $f_{k\epsilon}$ defined by

$$f_{k\epsilon}(x) = [(1-x)^{-1-k\epsilon}]_+ = -\frac{1}{k\epsilon}\,\delta(1-x) + \sum_{i=0}\frac{(-k\epsilon)^i}{i\,!}\,\mathcal{D}_i\,. \tag{3.3}$$

The α_s^n contributions $\mathcal{F}_n$ and $\mathcal{S}_n$ in eq. (3.2) exhibit poles in ϵ up to order ϵ^{-2n}. The corresponding bare structure function $F_n^{\rm b}$, on the other hand, only include terms up to ϵ^{-n}, as the higher divergences on the right-hand sides cancel for these inclusive quantities due to the Kinoshita-Lee-Nauenberg theorem [29, 30]. In fact, the complete cancellation already occurs at the level of the individual diagrams of ref. [15] for the forward Compton amplitude.

Once the products of lower-order quantities in eq. (3.2) have been subtracted from $F_n^{\rm b}$, the contribution of the n-loop form factor $\mathcal{F}_n$ can be extracted by performing the substitution

$$\mathcal{D}_0 \;\to\; \frac{1}{n\epsilon}\,\delta(1-x) - \sum_{i=1}\frac{(-n\epsilon)^i}{i\,!}\,\mathcal{D}_i \tag{3.4}$$

which eliminates, besides the +-distributions, the remaining $\delta(1-x)$ originating in the factor $f_{n\epsilon}$ of eq. (3.3) in the purely real part $\mathcal{S}_n$. However, as $\delta(1-x)$ enters $f_{n\epsilon}$ with a factor $1/\epsilon$, this extraction does not work at the highest order of ϵ kept in the calculation of $F_n^{\rm b}$. Hence, as stated above, the determination of the n-loop form factor $\mathcal{F}_n$ to order ϵ^k in this approach requires the calculation of the bare partonic structure function $F_n^{\rm b}$ to order ϵ^{k+1}.

In addition, the subtraction of the lower-order contributions $\mathcal{F}_l$ and $\mathcal{S}_l$ with $l < n$ in eq. (3.2) requires the extension of these quantities to higher orders in ϵ. Specifically, the first- and second-order quantities are required to order ϵ^3 and ϵ^1, respectively, for the extraction of the pole terms of the three-loop form factor. These functions have been determined from the calculation of F_1^{b} to order ϵ^4 and F_2^{b} to order ϵ^2. In fact, anticipating a future extension to the finite parts of the three-loop form factor $\mathcal{F}_3$, we have extended these calculations to one more power of ϵ, making use of the fact that the one- and two-loop integrals for the calculation of the structure functions were evaluated to order ϵ^5 and ϵ^3 anyway, see table 3 of ref. [15]. As a check of these new two-loop results (the one-loop quantities are known to all orders in ϵ anyway), a separate calculation of $\mathcal{F}_2$ has been performed to order ϵ^2 in the approach of refs. [5, 6]. The results for the corresponding seven diagrams are listed in appendix B.

To the accuracy in ϵ just discussed, the unrenormalized quark form factor reads, up to three loops in the notation of eq. (2.13),

$$\begin{aligned}
\mathcal{F}_1 = C_F\Bigg\{ & -2\frac{1}{\epsilon^2} - 3\frac{1}{\epsilon} - 8 + \zeta_2 + \epsilon\left(-16 + \frac{3}{2}\zeta_2 + \frac{14}{3}\zeta_3\right) + \\
& + \epsilon^2\left(-32 + 4\zeta_2 + 7\zeta_3 + \frac{47}{20}\zeta_2^2\right) \\
& + \epsilon^3\left(-64 + 8\zeta_2 + \frac{56}{3}\zeta_3 + \frac{141}{40}\zeta_2^2 - \frac{7}{3}\zeta_2\zeta_3 + \frac{62}{5}\zeta_5\right) + \\
& + \epsilon^4\left(-128 + 16\zeta_2 + \frac{112}{3}\zeta_3 + \frac{47}{5}\zeta_2^2 - \frac{7}{2}\zeta_2\zeta_3 + \frac{93}{5}\zeta_5 + \frac{949}{280}\zeta_2^3 - \frac{49}{9}\zeta_3^2\right)\Bigg\}, \quad (3.5)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_2 = C_F^2\Bigg\{ & 2\frac{1}{\epsilon^4} + 6\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{41}{2} - 2\zeta_2\right) + \frac{1}{\epsilon}\left(\frac{221}{4} - \frac{64}{3}\zeta_3\right) + \frac{1151}{8} + \frac{17}{2}\zeta_2 - \\
& - 58\zeta_3 - 13\zeta_2^2 + \epsilon\left(\frac{5741}{16} + \frac{213}{4}\zeta_2 - \frac{839}{3}\zeta_3 - \frac{171}{5}\zeta_2^2 + \frac{112}{3}\zeta_2\zeta_3 - \frac{184}{5}\zeta_5\right) + \\
& + \epsilon^2\Bigg(\frac{27911}{32} + \frac{1839}{8}\zeta_2 - \frac{6989}{6}\zeta_3 - \frac{3401}{20}\zeta_2^2 + 54\zeta_2\zeta_3 - \\
& \quad - \frac{462}{5}\zeta_5 + \frac{223}{5}\zeta_2^3 + \frac{2608}{9}\zeta_3^2\Bigg)\Bigg\} + \\
+ C_F C_A\Bigg\{ & -\frac{11}{6}\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{83}{9} + \zeta_2\right) + \frac{1}{\epsilon}\left(-\frac{4129}{108} - \frac{11}{6}\zeta_2 + 13\zeta_3\right) - \\
& - \frac{89173}{648} - \frac{119}{9}\zeta_2 + \frac{467}{9}\zeta_3 + \frac{44}{5}\zeta_2^2 + \\
& + \epsilon\left(-\frac{1775893}{3888} - \frac{6505}{108}\zeta_2 + \frac{6586}{27}\zeta_3 + \frac{1891}{60}\zeta_2^2 - \frac{89}{3}\zeta_2\zeta_3 + 51\zeta_5\right) + \\
& + \epsilon^2\Bigg(-\frac{33912061}{23328} - \frac{146197}{648}\zeta_2 + \frac{159949}{162}\zeta_3 + \frac{2639}{18}\zeta_2^2 - \\
& \quad - \frac{397}{9}\zeta_2\zeta_3 + \frac{3491}{15}\zeta_5 - \frac{809}{70}\zeta_2^3 - \frac{569}{3}\zeta_3^2\Bigg)\Bigg\} + \\
+ n_f C_F\Bigg\{ & \frac{1}{3}\frac{1}{\epsilon^3} + \frac{14}{9}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{353}{54} + \frac{1}{3}\zeta_2\right) + \frac{7541}{324} + \frac{14}{9}\zeta_2 - \frac{26}{9}\zeta_3 +
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\left(\frac{150125}{1944}+\frac{353}{54}\zeta_2-\frac{364}{27}\zeta_3-\frac{41}{30}\zeta_2^2\right)+\\
&+\epsilon^2\left(\frac{2877653}{11664}-\frac{26}{9}\zeta_2\zeta_3+\frac{7541}{324}\zeta_2-\frac{4589}{81}\zeta_3-\frac{287}{45}\zeta_2^2-\frac{242}{15}\zeta_5\right)\Bigg\}, \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_3 = C_F^3\Bigg\{&-\frac{4}{3}\frac{1}{\epsilon^6}-6\frac{1}{\epsilon^5}+\frac{1}{\epsilon^4}\left(-25+2\zeta_2\right)+\frac{1}{\epsilon^3}\left(-83-3\zeta_2+\frac{100}{3}\zeta_3\right)+\\
&+\frac{1}{\epsilon^2}\left(-\frac{515}{2}-\frac{77}{2}\zeta_2+138\zeta_3+\frac{213}{10}\zeta_2^2\right)+\\
&+\frac{1}{\epsilon}\left(-\frac{9073}{12}-\frac{467}{2}\zeta_2+\frac{2119}{3}\zeta_3+\frac{1461}{20}\zeta_2^2-\frac{214}{3}\zeta_2\zeta_3+\frac{644}{5}\zeta_5\right)\Bigg\}+\\
+C_F^2C_A\Bigg\{&\frac{11}{3}\frac{1}{\epsilon^5}+\frac{1}{\epsilon^4}\left(\frac{431}{18}-2\zeta_2\right)+\frac{1}{\epsilon^3}\left(\frac{6415}{54}-\frac{7}{6}\zeta_2-26\zeta_3\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{79277}{162}+\frac{1487}{36}\zeta_2-\frac{83}{5}\zeta_2^2-210\zeta_3\right)+\\
&+\frac{1}{\epsilon}\left(\frac{1773839}{972}+\frac{38623}{108}\zeta_2-\frac{6703}{6}\zeta_3-\frac{9839}{72}\zeta_2^2+\frac{215}{3}\zeta_2\zeta_3-142\zeta_5\right)\Bigg\}+\\
+C_FC_A^2\Bigg\{&-\frac{242}{81}\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\left(-\frac{6521}{243}+\frac{88}{27}\zeta_2\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{40289}{243}-\frac{553}{81}\zeta_2+\frac{1672}{27}\zeta_3-\frac{88}{45}\zeta_2^2\right)+\\
&+\frac{1}{\epsilon}\left(-\frac{1870564}{2187}-\frac{68497}{486}\zeta_2+\frac{12106}{27}\zeta_3+\frac{802}{15}\zeta_2^2-\frac{88}{9}\zeta_2\zeta_3-\frac{136}{3}\zeta_5\right)\Bigg\}+\\
+n_fC_F^2\Bigg\{&-\frac{2}{3}\frac{1}{\epsilon^5}-\frac{37}{9}\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\left(-\frac{545}{27}-\frac{1}{3}\zeta_2\right)+\frac{1}{\epsilon^2}\left(-\frac{6499}{81}-\frac{133}{18}\zeta_2+\frac{146}{9}\zeta_3\right)+\\
&+\frac{1}{\epsilon}\left(-\frac{138865}{486}-\frac{2849}{54}\zeta_2+\frac{2557}{27}\zeta_3+\frac{337}{36}\zeta_2^2\right)\Bigg\}+\\
+n_fC_FC_A\Bigg\{&\frac{88}{81}\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{2254}{243}-\frac{16}{27}\zeta_2\right)+\frac{1}{\epsilon^2}\left(\frac{13679}{243}+\frac{316}{81}\zeta_2-\frac{256}{27}\zeta_3\right)+\\
&+\frac{1}{\epsilon}\left(\frac{623987}{2187}+\frac{11027}{243}\zeta_2-\frac{6436}{81}\zeta_3-\frac{44}{5}\zeta_2^2\right)\Bigg\}+ \qquad (3.7)\\
+n_f^2C_F\Bigg\{&-\frac{8}{81}\frac{1}{\epsilon^4}-\frac{188}{243}\frac{1}{\epsilon^3}+\frac{1}{\epsilon^2}\left(-\frac{124}{27}-\frac{4}{9}\zeta_2\right)+\frac{1}{\epsilon}\left(-\frac{49900}{2187}-\frac{94}{27}\zeta_2+\frac{136}{81}\zeta_3\right)\Bigg\}.
\end{aligned}
$$

Here n_f stands for the number of effectively massless quark flavours, C_F and C_A are the usual QCD colour factors, $C_F = 4/3$ and $C_A = 3$, and the values of Riemann's zeta function are denoted by ζ_n.

Eq. (3.7) and the ϵ^1 and ϵ^2 parts of eq. (3.6) are new results of this article. The four highest $1/\epsilon$ poles of the three-loop form factor $\mathcal{F}_3$ provide the first complete verification of the resummation of the next-to-leading contributions. With the anomalous dimensions (2.5) known up to A_3, the remaining two poles are sufficient to fix the NNL contributions to the function G in eq. (2.9). Especially, we can derive the first ($\epsilon = 0$) term of the third-order function $G_3(\epsilon)$.

Before we turn to these results we recall, for completeness, the known coefficients of the cusp anomalous dimension $A(a_s)$. The results for A_1 and A_2,

$$A_1 = 4\,C_F\,, \qquad A_2 = 8\,C_F C_A\left(\frac{67}{18} - \zeta_2\right) + 8\,C_F n_f\left(-\frac{5}{9}\right), \tag{3.8}$$

have been known for a long time [31]. The recently completed expression for A_3 reads [17]

$$\begin{aligned} A_3 \,=\, & 16\,C_F C_A^2\left(\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2\right) + 16\,C_F^2 n_f\left(-\frac{55}{24} + 2\,\zeta_3\right) + \\ & +16\,C_F C_A n_f\left(-\frac{209}{108} + \frac{10}{9}\,\zeta_2 - \frac{7}{3}\,\zeta_3\right) + 16\,C_F n_f^2\left(-\frac{1}{27}\right). \end{aligned} \tag{3.9}$$

See refs. [32 − 34] for previous partial results on the n_f-contributions. Very recently the ζ_2^2 term in eq. (3.9) has been confirmed in ref. [19], see the discussion at the end of section 4.

Inserting eqs. (3.8) and (3.9) into the resummation relations (2.14)–(2.16) and comparing to the explicit results (3.5)–(3.7), we obtain the following perturbative expansion of eq. (2.9) at $\mu^2 = Q^2$:

$$\begin{aligned} G_1 \,=\, & 6C_F + \epsilon C_F(16 - 2\zeta_2) + \epsilon^2 C_F\left(32 - 3\zeta_2 - \frac{28}{3}\zeta_3\right) + \\ & +\epsilon^3 C_F\left(64 - 8\zeta_2 - 14\zeta_3 - \frac{47}{10}\zeta_2^2\right) + \\ & +\epsilon^4 C_F\left(128 - 16\zeta_2 - \frac{112}{3}\zeta_3 - \frac{141}{20}\zeta_2^2 + \frac{14}{3}\zeta_2\zeta_3 - \frac{124}{5}\zeta_5\right) + \\ & +\epsilon^5 C_F\left(256 - 32\zeta_2 - \frac{224}{3}\zeta_3 - \frac{94}{5}\zeta_2^2 + 7\zeta_2\zeta_3 - \frac{186}{5}\zeta_5 - \frac{949}{140}\zeta_2^3 + \frac{98}{9}\zeta_3^2\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} G_2 \,=\, & C_F^2(3 - 24\zeta_2 + 48\zeta_3) + C_F C_A\left(\frac{2545}{27} + \frac{44}{3}\zeta_2 - 52\zeta_3\right) + \\ & +C_F n_f\left(-\frac{418}{27} - \frac{8}{3}\zeta_2\right) + \epsilon C_F^2\left(\frac{1}{2} - 116\zeta_2 + 120\zeta_3 + \frac{176}{5}\zeta_2^2\right) + \\ & +\epsilon C_F C_A\left(\frac{70165}{162} + \frac{575}{9}\zeta_2 - \frac{520}{3}\zeta_3 - \frac{176}{5}\zeta_2^2\right) + \\ & +\epsilon C_F n_f\left(-\frac{5813}{81} - \frac{74}{9}\zeta_2 + \frac{16}{3}\zeta_3\right) + \\ & +\epsilon^2 C_F^2\left(-\frac{109}{4} - 437\zeta_2 + 736\zeta_3 + \frac{432}{5}\zeta_2^2 - 112\zeta_2\zeta_3 + 48\zeta_5\right) + \\ & +\epsilon^2 C_F C_A\left(\frac{1547797}{972} + \frac{7297}{27}\zeta_2 - \frac{24958}{27}\zeta_3 - \frac{653}{6}\zeta_2^2 + \frac{356}{3}\zeta_2\zeta_3 - 204\zeta_5\right) + \\ & +\epsilon^2 n_f C_F\left(-\frac{129389}{486} - \frac{850}{27}\zeta_2 + \frac{1204}{27}\zeta_3 + \frac{7}{3}\zeta_2^2\right) \\ & +\epsilon^3 C_F^2\left(-\frac{1287}{8} - \frac{2991}{2}\zeta_2 + 3614\zeta_3 + 508\zeta_2^2 - \right. \\ & \left. \quad - 104\zeta_2\zeta_3 + 72\zeta_5 - \frac{6864}{35}\zeta_2^3 - 1072\zeta_3^2\right) + \end{aligned}$$

$$+\epsilon^3 C_F C_A\left(\frac{31174909}{5832}+\frac{155701}{162}\zeta_2-\frac{308810}{81}\zeta_3-\frac{100907}{180}\zeta_2^2+\right.$$
$$\left.+\frac{478}{3}\zeta_2\zeta_3-840\zeta_5+\frac{1618}{35}\zeta_2^3+\frac{2276}{3}\zeta_3^2\right)+$$
$$+\epsilon^3 n_f C_F\left(-\frac{2628821}{2916}-\frac{8405}{81}\zeta_2+\frac{16340}{81}\zeta_3+\frac{1873}{90}\zeta_2^2+\frac{44}{3}\zeta_2\zeta_3+48\zeta_5\right), \quad (3.11)$$

$$\begin{aligned} G_3 = {} & C_F^3\left(29+36\zeta_2+136\zeta_3+\frac{576}{5}\zeta_2^2-64\zeta_2\zeta_3-480\zeta_5\right)+\\ & +C_F^2 C_A\left(\frac{232}{3}-\frac{2096}{3}\zeta_2+\frac{3008}{3}\zeta_3-\frac{8}{3}\zeta_2^2+32\zeta_2\zeta_3+240\zeta_5\right)+\\ & +C_F C_A^2\left(\frac{1045955}{729}+\frac{34732}{81}\zeta_2-\frac{34928}{27}\zeta_3-\frac{188}{3}\zeta_2^2+\frac{176}{3}\zeta_2\zeta_3+272\zeta_5\right)+\\ & +n_f C_F^2\left(-\frac{3826}{27}+\frac{296}{3}\zeta_2-\frac{1232}{9}\zeta_3+\frac{208}{15}\zeta_2^2\right)+\\ & +n_f C_F C_A\left(-\frac{309838}{729}-\frac{11728}{81}\zeta_2+\frac{1448}{9}\zeta_3+\frac{88}{15}\zeta_2^2\right)+\\ & +n_f^2 C_F\left(\frac{19676}{729}+\frac{304}{27}\zeta_2+\frac{32}{27}\zeta_3\right). \end{aligned} \quad (3.12)$$

As discussed at the end of section 2, these results are sufficient to fix the next-to-next-to-leading contributions, i.e., the six highest poles in ϵ, to all orders in the strong coupling. In fact, in view of a future extension of G_3 to order ϵ, the first- and second-order results (3.10) and (3.11) already transcend this accuracy by one power in ϵ.

We close this section by a brief discussion of our three-loop results (3.7) and (3.12). The former result for the $1/\epsilon$ poles of the quark form factor in massless QCD is not directly applicable to any physical process. For use in cross section calculations such as $e^+e^- \to 2$ jets at the next-to-next-to-next-to-leading order ($\mathrm{N^3LO}$), one would need the finite contribution to $\mathcal{F}_3$ as well. However, the resulting leading term (3.12) of G_3 is of immediate interest for predictions of the pole structure of QCD amplitudes at higher orders [13, 14] generalizing Catani's NNLO formula [12]. For the four-quark amplitude at N^3LO, qq $\to$ qq, for instance, an explicit prediction has been derived in ref. [13], for which eqs. (3.7) and (3.12) now provide the last missing piece of information.

4. The time-like case and non-QCD applications

So far, our discussion has been restricted to space-like photon momenta, $q^2 = -Q^2 < 0$. The modifications for the time-like case $q^2 > 0$ are obtained by analytic continuation. For the resummed quark form factor in eq. (2.10) this continuation has been discussed in ref. [9], while the finite-order expansions (2.13) are transferred to $q^2 > 0$ according to [5]

$$\left(\frac{-q^2}{\mu^2}\right)^{-l\epsilon} = \left(\frac{q^2}{\mu^2}\right)^{-l\epsilon}\left(\frac{\Gamma(1-l\epsilon)\Gamma(1+l\epsilon)}{\Gamma(1-2l\epsilon)\Gamma(1+2l\epsilon)} - \mathrm{i}\frac{\pi l\epsilon}{\Gamma(1-l\epsilon)\Gamma(1+l\epsilon)}\right). \quad (4.1)$$

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Of particular interest is the absolute ratio $|\mathcal{F}(q^2)/\mathcal{F}(-q^2)|$ of the renormalized time-like and space-like form factors. This quantity is infrared finite and directly enters the cross section for Drell-Yan lepton pair production in hadronic collisions. Transforming eqs. (2.14)–(2.17) back to the renormalized quantities using eqs. (2.11) and (2.12), and then employing the analytic continuation (4.1) we obtain the expansion

$$
\begin{aligned}
\left|\frac{\mathcal{F}(q^2)}{\mathcal{F}(-q^2)}\right|^2 &= 1 + a_{\rm s}\{3\zeta_2 A_1\} + a_{\rm s}^2\left\{\frac{9}{2}\zeta_2^2 A_1^2 + 3\zeta_2(\beta_0 G_1 + A_2)\right\} + \\
&\quad + a_{\rm s}^3\left\{\frac{9}{2}\zeta_2^3 A_1^3 + 3\zeta_2^2 A_1(3\beta_0 G_1 - \beta_0^2 + 3A_2) + 3\zeta_2(A_3 + \beta_1 G_1 + 2\beta_0 G_2)\right\} + \\
&\quad + a_{\rm s}^4\left\{\frac{27}{8}\zeta_2^4 A_1^4 + \frac{9}{2}\zeta_2^3 A_1^2(3\beta_0 G_1 - 2\beta_0^2 + 3A_2) + \right.\\
&\qquad + \frac{3}{2}\zeta_2^2(-6\beta_0^2 A_2 + 3\beta_0^2 G_1^2 + 3A_2^2 + 12\beta_0 A_1 G_2 + 6A_1 A_3 + 6\beta_1 A_1 G_1 - \\
&\qquad\quad - 5\beta_0\beta_1 A_1 + 6\beta_0 A_2 G_1 - 6\beta_0^3 G_1) + \\
&\qquad\quad \left. + 3\zeta_2(A_4 + \beta_2 G_1 + 3\beta_0 G_3 + 2\beta_1 G_2)\right\} + \mathcal{O}(a_{\rm s}^5)
\end{aligned} \tag{4.2}
$$

in terms of the couplings $a_{\rm s}(q^2) = a_{\rm s}(-q^2) = a_{\rm s}$. Note that, since this ratio is infrared finite, only the $\epsilon = 0$ parts of the coefficients G_i enter eq. (4.2). Consequently, all terms contributing at the fourth order are now known, with the exception of the four-loop cusp anomalous dimension A_4 of which only the small n_f^3 contribution has been derived so far [32].

The effect of A_4 is expected to be small, therefore we can nevertheless evaluate the ratio (4.2) also numerically up to the fourth order, employing the [1/1] Padé estimate of ref. [16],

$$
A_{\rm q,4} \approx 7849, 4313, 1553 \quad \text{for } n_f = 3, 4, 5\,, \tag{4.3}
$$

to which we assign a conservative 50% uncertainty. Switching back to the strong coupling $\alpha_{\rm s} = 4\pi\, a_{\rm s}$ at the scale q^2 as the expansion parameter, eq. (4.2) for $n_f = 4$ yields the numerical expansion

$$
\left|\frac{\mathcal{F}(q^2)}{\mathcal{F}(-q^2)}\right|^2 = 1 + 2.094\,\alpha_{\rm s} + 5.613\,\alpha_{\rm s}^2 + 15.70\,\alpha_{\rm s}^3 + (48.63 \pm 0.43)\,\alpha_{\rm s}^4\,. \tag{4.4}
$$

This result does not look like a nicely converging expansion, but so far does not exhibit a clear factorial growth of the higher-order coefficients either. As already pointed out in ref. [9], the only genuine l-loop contribution at order $\alpha_{\rm s}^l$ is given by the anomalous dimension A_l, which in eq. (4.4) contributes 24%, 7% and (2 ± 1)% of the total coefficient at the second, third and fourth order, respectively. On the other hand, the contributions of the quantities G_{l-1} at order $\alpha_{\rm s}^l$ are large, amounting to 37%, 41%, 50% at $l = 2$, 3, 4. Consequently, the higher-order ($l \geq 5$) terms in eq. (4.4) cannot be predicted quantitatively at this point.

Exponentiations like eq. (2.10) for the form factor $\mathcal{F}$ have also been studied for electroweak interactions [20], where a fermion or gauge-boson mass m acts as a regulator for collinear or infrared singularities. Of course, both the counter-term function K in eq. (2.4) and the lower integration limit in eq. (2.10) are modified in this case, as they depend on the infrared sector of the theory. However, the leading ($\epsilon = 0$) term of the function G in eq. (2.3) is independent of the regulator at each order in the coupling constant. This contribution entirely originates in the so-called hard region in an expansion of the loop integrals in different regions [35]. In this region all loop momenta are of order Q, effectively leading to the massless case considered in eqs. (2.16) and (3.7).

This 'universality' implies, for instance, that eq. (3.12) provides a prediction for the coefficient $C_F^3 \ln(Q^2/m^2)$ in the three-loop quantity $\mathcal{F}_3$ for an abelian gauge theory with fermion masses like Quantum Electrodynamics (QED) [36] after the usual identification of the colour factors. For QED, e.g., one has $C_F = 1$, $C_A = 0$ and $T_f = 1$ instead of our QCD convention $T_f\, n_f = n_f/2$.

Another interesting implication of eq. (3.7) arises for maximally supersymmetric Yang-Mills theory (MSYM), i.e., Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry in four dimensions. QCD results may be carried over to this theory using the inspired observation [37] that the MSYM results can be obtained from the contributions of leading transcendentality in QCD. This procedure has been applied to the QCD results for the three-loop anomalous dimensions of spin N of leading-twist operators [17, 18], which were employed to extract corresponding quantities in MSYM [37]. Strikingly enough, the resulting MSYM anomalous dimensions completely agree with predictions based on integrability for the planar three-loop contribution to the dilatation operator [38]. This agreement has been checked up to spin $N = 8$ in ref. [39] and is now established up to $N = 70$ [40] (for a review see also ref. [41]).

Although no formal proof exists for the procedure of ref. [37], it has recently been used in reverse, namely to predict terms of highest transcendentality in the QCD form factor. Based on studies of planar amplitudes in MSYM at three loops [19], where an interesting pattern of iteration for the four-point amplitude has been found, both the coefficients $A_l|_{\mathrm{MSYM}}$ and the leading contribution to $G_l|_{\mathrm{MSYM}}$ have been determined for $l \leq 3$. Our new result for the three-loop form factor $\mathcal{F}_3$ in eq. (3.7) and for coefficient G_3 in eq. (3.12) puts us in a position to check this part of ref. [19] and thereby provide further evidence on the procedure of ref. [37].

The only transcendental numbers entering the results for the form factor are the values ζ_n of Riemann's zeta function. Hence the procedure of ref. [37] implies that, at each order in α_{s}, one keeps only the highest terms ζ_n and $\zeta_i\, \zeta_j$ with $i+j = n$. After the SYM identification $C_A = C_F = n_c$ (terms with n_f do not contribute at the highest transcendentality), eqs. (3.8) and (3.9) lead to

$$A_1\Big|_{\mathrm{MSYM}} = 4\, n_c\,, \qquad A_2\Big|_{\mathrm{MSYM}} = -8\, \zeta_2 n_c^2\,, \qquad A_3\Big|_{\mathrm{MSYM}} = \frac{176}{5}\, \zeta_2^2 n_c^3\,. \tag{4.5}$$

Correspondingly, eqs. (3.10)–(3.12) result in

$$G_1\Big|_{\mathrm{MSYM}} = 0\,, \qquad G_2\Big|_{\mathrm{MSYM}} = -4\, \zeta_3 n_c^2\,, \qquad G_3\Big|_{\mathrm{MSYM}} = \frac{80}{3}\, \zeta_2 \zeta_3 n_c^3 + 32\, \zeta_5 n_c^3\,. \tag{4.6}$$

Both relations agree with the results of ref. [19], and hence with the prescription of ref. [37].

5. Summary

We have derived new higher-order QCD results for the electromagnetic form factor of on-shell massless quarks. Specifically, we have extracted all third-order $1/\epsilon$ pole terms in dimensional regularization from our recent computation of the three-loop coefficient functions for inclusive deep-inelastic scattering [15], supplemented by a higher-ϵ extension of the two-loop contributions. These results, together with our extension of the resummation of the form factor to the next-to-next-to-leading contributions, fix the six highest $1/\epsilon$ poles to all orders. As an example, we have provided the explicit expression for the coefficients of $\epsilon^{-8} \ldots \epsilon^{-3}$ at four loops.

While the pole terms of the form factor alone are not sufficient for use in other three-loop calculations like $e^+e^- \to 2$ jets, they do have immediate theoretical applications both for the infrared structure of higher-order QCD amplitudes and for other gauge theories such as QED and $\mathcal{N} = 4$ Super-Yang-Mills theory, where our results confirm a recent corresponding calculation in ref. [19]. Moreover, our present results are sufficient (up to a numerically irrelevant uncertainty due to the unknown four-loop cusp anomalous dimension) for extending the finite absolute ratio of the time-like and space-like form factors, which directly enters the description of the Drell-Yan process, to the fourth order in $\alpha_{\rm s}$.

We close by noting that the computation of the finite part of the three-loop quark form factor $\mathcal{F}_3$ by an extension of the techniques employed in this article is feasible.

Note added in proof. Shortly after submission of the present article, ref. [49] appeared. In that paper the two-loop quark and gluon form factors are calculated in dimensional regularisation, and explicit expressions to all orders in ϵ are presented. This calculation confirms our result for $\mathcal{F}_2$ in eqs. (3.6) and (B.1).

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A. Integrals for form factor resummation

Here we give some details for the determination of the loop-expanded form factor from eq. (2.10). Useful auxiliary relations are

$$\frac{1}{(1-x)^{n-\epsilon}} = \sum_{i=0}^{\infty} \frac{\Gamma(n-\epsilon+i)}{\Gamma(n-\epsilon)} \, \frac{x^i}{i!} \,, \tag{A.1}$$

$$\frac{\ln^k(1-x)}{(1-x)^{n-\epsilon}} = \left(\frac{\partial}{\partial\epsilon}\right)^k \frac{1}{(1-x)^{n-\epsilon}} \,, \tag{A.2}$$

where eq. (A.1) holds for $|x| < 1$. The expansion of the Gamma function in powers of ϵ for positive integers n reads

$$\begin{aligned}\frac{\Gamma(n+1+\epsilon)}{n!\,\Gamma(1+\epsilon)} &= 1 + \epsilon S_1(n) + \epsilon^2(S_{1,1}(n) - S_2(n)) + \epsilon^3(S_{1,1,1}(n) - S_{1,2}(n)) - S_{2,1}(n) + \\ &\quad + S_3(n) + \epsilon^4(S_{1,1,1,1}(n) - S_{1,1,2}(n) - S_{1,2,1}(n) + S_{1,3}(n) - S_{2,1,1}(n) + \\ &\quad + S_{2,2}(n) + S_{3,1}(n) - S_4(n)) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{A.3}$$

with $S_{m_1,\ldots,m_k}(N)$ denoting the harmonic sums [26]. Finally integrals of the following types occur:

$$\int \frac{d\lambda}{\lambda}\,\lambda^{-n\epsilon} = -\frac{1}{n\epsilon}\lambda^{-n\epsilon}\,, \tag{A.4}$$

$$\begin{aligned}\int \frac{d\lambda}{\lambda}\left(\lambda^{-\epsilon}-1\right)^n &= -\frac{1}{\epsilon}(-1)^n \sum_{j=1}^{n} \frac{(-1)^j}{j}\left(\lambda^{-\epsilon}-1\right)^j \\ &\quad -\frac{1}{\epsilon}(-1)^n \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^j}{j} + (-1)^n \ln\lambda\,, \end{aligned} \tag{A.5}$$

$$\int \frac{d\lambda}{\lambda}\,\lambda^{-\epsilon}\left(\lambda^{-\epsilon}-1\right)^n = -\frac{1}{\epsilon}\frac{1}{n+1}\left(\lambda^{-\epsilon}-1\right)^{n+1}, \tag{A.6}$$

$$\int \frac{d\lambda}{\lambda}\,\lambda^{-2\epsilon}\left(\lambda^{-\epsilon}-1\right)^n = -\frac{1}{\epsilon}\left(\frac{1}{n+1}+\lambda^{-\epsilon}\right)\frac{1}{n+2}\left(\lambda^{-\epsilon}-1\right)^{n+1}, \tag{A.7}$$

and so on, where $n > 0$. The integration over λ and ξ in eq. (2.10) leads to double sums which are readily evaluated to any finite order in α_s. Also all-order analytical results for the exponent, cf. ref. [10], can be obtained along these lines by employing the algorithms for the evaluation of nested sums [27].

B. Feynman diagrams for two-loop form factor

Finally we present the results for the individual Feynman diagrams, displayed in figure 1, which contribute to the two-loop form factor in the approach (and notation) of refs. [5, 6]. The diagrams add up to the bare quark form factor in eq. (3.6) according to

$$\mathcal{F}_2 = 2S + QL + GL + 2QV + 2GV + C + L\,. \tag{B.1}$$

Note that both the normalization in eq. (2.13), where we have pulled out the factor $(Q^2/\mu^2)^{-2\epsilon}$, and our convention for ϵ are different from those in ref. [6]. The results for the individual diagrams to order ϵ^2 are given by

$$\begin{aligned}S = C_F^2\Bigg\{ &\frac{1}{\epsilon^3} + \frac{7}{2}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{53}{4} - \zeta_2\right) + \frac{355}{8} - \frac{7}{2}\zeta_2 - \frac{32}{3}\zeta_3 + \\ &+ \epsilon\left(\frac{2281}{16} - \frac{53}{4}\zeta_2 - \frac{112}{3}\zeta_3 - \frac{57}{10}\zeta_2^2\right) \\ &+ \epsilon^2\left(\frac{14299}{32} - \frac{355}{8}\zeta_2 - \frac{424}{3}\zeta_3 - \frac{399}{20}\zeta_2^2 + \frac{32}{3}\zeta_2\zeta_3 - \frac{272}{5}\zeta_5\right)\Bigg\}\,, \end{aligned} \tag{B.2}$$

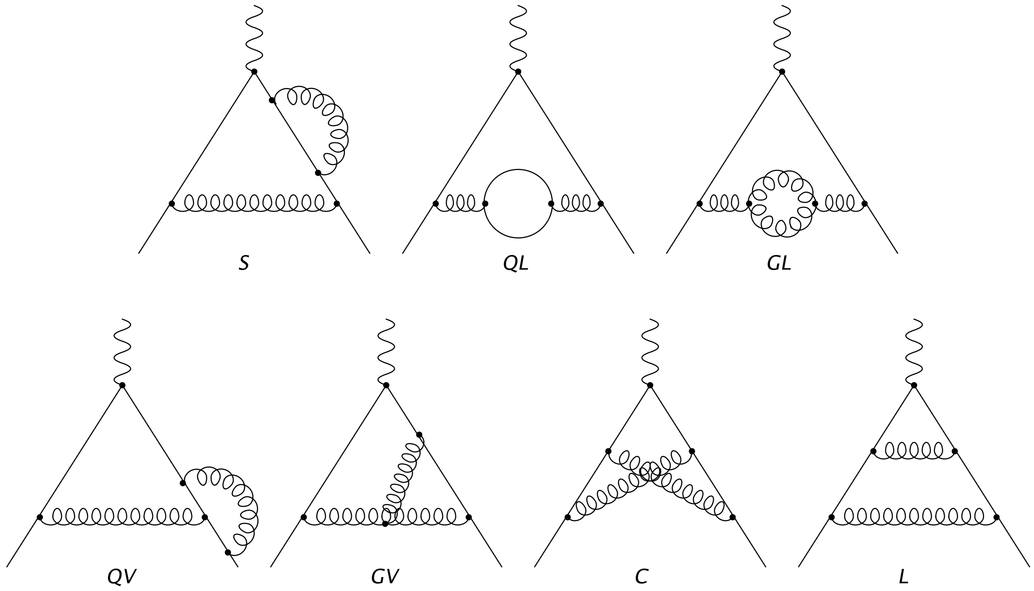

Figure 1: The Feynman diagrams contributing to the two-loop quark form factor in the notation of ref. [6].

$$\begin{aligned}
QL &= C_F n_f \left\{ \frac{1}{3}\frac{1}{\epsilon^3} + \frac{14}{9}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{353}{54} + \frac{1}{3}\zeta_2\right) + \frac{7541}{324} + \frac{14}{9}\zeta_2 - \frac{26}{9}\zeta_3 + \right. \\
&\quad + \epsilon\left(\frac{150125}{1944} + \frac{353}{54}\zeta_2 - \frac{364}{27}\zeta_3 - \frac{41}{30}\zeta_2^2\right) + \\
&\quad \left. + \epsilon^2\left(\frac{2877653}{11664} + \frac{7541}{324}\zeta_2 - \frac{4589}{81}\zeta_3 - \frac{287}{45}\zeta_2^2 - \frac{26}{9}\zeta_2\zeta_3 - \frac{242}{15}\zeta_5\right)\right\}, \qquad \text{(B.3)} \\
GL &= C_F C_A \left\{ -\frac{5}{6}\frac{1}{\epsilon^3} - \frac{38}{9}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{1969}{108} - \frac{5}{6}\zeta_2\right) - \frac{43165}{648} - \frac{38}{9}\zeta_2 + \frac{65}{9}\zeta_3 + \right. \\
&\quad + \epsilon\left(-\frac{873877}{3888} - \frac{1969}{108}\zeta_2 + \frac{988}{27}\zeta_3 + \frac{41}{12}\zeta_2^2\right) + \qquad \text{(B.4)} \\
&\quad \left. + \epsilon^2\left(-\frac{16929277}{23328} - \frac{43165}{648}\zeta_2 + \frac{25597}{162}\zeta_3 + \frac{779}{45}\zeta_2^2 + \frac{65}{9}\zeta_2\zeta_3 + \frac{121}{3}\zeta_5\right)\right\}, \\
QV &= C_F\left(C_F - \frac{C_A}{2}\right)\left\{ -\frac{1}{\epsilon^3} - \frac{1}{\epsilon^2}\left(\frac{11}{2} - 2\zeta_2\right) + \frac{1}{\epsilon}\left(-\frac{109}{4} + 10\zeta_2 + 2\zeta_3\right) - \right. \\
&\quad - \frac{911}{8} + \frac{91}{2}\zeta_2 + \frac{59}{3}\zeta_3 + \frac{8}{5}\zeta_2^2 + \\
&\quad + \epsilon\left(-\frac{6957}{16} + \frac{689}{4}\zeta_2 + \frac{296}{3}\zeta_3 + \frac{129}{10}\zeta_2^2 - \frac{58}{3}\zeta_2\zeta_3 + 6\zeta_5\right) + \\
&\quad + \epsilon^2\left(-\frac{49639}{32} + \frac{4843}{8}\zeta_2 + \frac{1307}{3}\zeta_3 + \frac{1267}{20}\zeta_2^2 - \frac{293}{3}\zeta_2\zeta_3 + \right. \\
&\quad \left.\left. + \frac{407}{5}\zeta_5 - \frac{281}{35}\zeta_2^3 - \frac{58}{3}\zeta_3^2\right)\right\}, \qquad \text{(B.5)} \\
GV &= C_F C_A \left\{ \frac{1}{4}\frac{1}{\epsilon^4} + \frac{1}{\epsilon^2}\left(-\frac{5}{4} - \frac{1}{4}\zeta_2\right) + \frac{1}{\epsilon}\left(-\frac{73}{8} + \frac{1}{2}\zeta_2 - \frac{8}{3}\zeta_3\right) - \right.
\end{aligned}$$

$$
\begin{aligned}
& -\frac{663}{16}+\frac{15}{4}\zeta_2+\frac{1}{2}\zeta_3-\frac{57}{40}\zeta_2^2 \\
& +\epsilon\Big(-\frac{5093}{32}+\frac{159}{8}\zeta_2+\frac{95}{6}\zeta_3+\frac{2}{5}\zeta_2^2+\frac{8}{3}\zeta_2\zeta_3-\frac{68}{5}\zeta_5\Big)+ \\
& +\epsilon^2\Big(-\frac{35887}{64}+\frac{1289}{16}\zeta_2+\frac{1297}{12}\zeta_3+\frac{73}{8}\zeta_2^2-\frac{29}{6}\zeta_2\zeta_3+ \\
& \quad +\frac{3}{2}\zeta_5-\frac{1343}{280}\zeta_2^3+\frac{128}{9}\zeta_3^2\Big)\Big\}, \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
C = C_F(C_F-\frac{C_A}{2})\Big\{ & \frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\frac{1}{\epsilon^2}(16-7\zeta_2)+\frac{1}{\epsilon}\Big(58-16\zeta_2-\frac{122}{3}\zeta_3\Big)+ \\
& +204-58\zeta_2-\frac{380}{3}\zeta_3-\frac{53}{2}\zeta_2^2+ \\
& +\epsilon\Big(697-181\zeta_2-\frac{1646}{3}\zeta_3-\frac{402}{5}\zeta_2^2+\frac{326}{3}\zeta_2\zeta_3-\frac{842}{5}\zeta_5\Big)+ \\
& +\epsilon^2\Big(\frac{4631}{2}-\frac{1141}{2}\zeta_2-\frac{6293}{3}\zeta_3-\frac{1744}{5}\zeta_2^2+\frac{836}{3}\zeta_2\zeta_3- \\
& \quad -\frac{2708}{5}\zeta_5+\frac{1399}{70}\zeta_2^3+\frac{4274}{9}\zeta_3^2\Big)\Big\}, \qquad \text{(B.7)}
\end{aligned}
$$

$$
\begin{aligned}
L = C_F^2\Big\{ & \frac{1}{\epsilon^4}+\frac{2}{\epsilon^3}+\frac{1}{\epsilon^2}\Big(\frac{17}{2}+\zeta_2\Big)+\frac{1}{\epsilon}\Big(\frac{101}{4}-2\zeta_2+\frac{46}{3}\zeta_3\Big)+ \\
& +\frac{631}{8}-\frac{35}{2}\zeta_2+\frac{152}{3}\zeta_3+\frac{103}{10}\zeta_2^2 \\
& +\epsilon\Big(\frac{3941}{16}-\frac{335}{4}\zeta_2+\frac{439}{3}\zeta_3+\frac{159}{5}\zeta_2^2-\frac{98}{3}\zeta_2\zeta_3+\frac{598}{5}\zeta_5\Big)+ \\
& +\epsilon^2\Big(\frac{24495}{32}-\frac{2573}{8}\zeta_2+\frac{2065}{6}\zeta_3+\frac{1839}{20}\zeta_2^2-\frac{152}{3}\zeta_2\zeta_3+ \\
& \quad +\frac{1976}{5}\zeta_5+\frac{2847}{70}\zeta_2^3-\frac{1318}{9}\zeta_3^2\Big)\Big\}. \qquad \text{(B.8)}
\end{aligned}
$$

The loop integrations have been reduced with integration-by-parts identities [44, 45] to so-called master integrals. This step has been automatized in ref. [46]. All master integrals except the basic non-planar triangle can be expressed in terms of Gamma functions, thus they can be readily expanded to any order in ϵ. The non-planar triangle (see, e.g., ref. [47]) can be written as a double sum over Gamma functions. After expansion, the sums can be solved in terms of the Riemann zeta function to any order in ϵ using the algorithms for harmonic sums [26] coded, as all our symbolic manipulations, in Form [48].

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